Interval Estimates for Probabilities of Non-Perforation Using a Generalized Pivotal Quantity

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Abstract

A generalized pivotal quantity is developed that yields confidence intervals for the cumulative distribution function (CDF) at a specific value when the underlying distribution is assumed to be normal. This problem is similar to the development of a tolerance interval, and, unsurprisingly, its solution involves the non-central t distribution. Generalized confidence bands for a normal CDF follow easily. Military applications include vulnerability and lethality assessment, for example, interval estimation for the probability of non-perforation against homogeneous armored targets.

Introduction

An engineer at the U.S. Army Research Laboratory at Aberdeen Proving Ground approached the author in the spring of 2004 requesting help in estimating the probability that a projectile would not penetrate beyond the thickness of an armor plate. That is, the client wanted an estimate for the probability of non-perforation. This estimate was to be obtained from sample data of depths of penetration into homogeneous armor of "infinite" thickness.

To model this phenomena, one could let X be defined as the penetration depth of a random projectile and assume that X is a normally distributed random variable with mean μ and variance σ^2. If x_0 is the thickness of the plate for which the probability of non-perforation estimate is desired, then the client seeks an estimate for $P(X < x_0)$, or equivalently $\Phi\left(\frac{x_0 - \mu}{\sigma}\right)$, where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

Using the sample mean and the sample standard deviation from the observations, the plug-in estimate, $\Phi\left(\frac{x_0 - \bar{x}}{s}\right)$, serves as an adequate point estimate for the probability of non-perforation. Of course, point estimates yield no information on the error of estimation. What we'd prefer to report is a confidence interval for $\Phi\left(\frac{x_0 - \mu}{\sigma}\right)$ so that one can say, e.g., "… with 95% confidence, the probability of non-perforation is between some lower confidence limit (LCL) and upper confidence limit (UCL)." Interval estimation of $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$ is not a new problem – see Owen & Hua (1977), Odeh & Owen (1980), Hahn & Meeker (1990), or Patel & Read (1996). However, in this paper we will examine this problem from the perspective of generalized confidence intervals, a technique that allows one to obtain confidence intervals for *any* function of μ and σ^2.

Classical and Generalized Confidence Intervals

In the ensuing discussion, it is imperative that we make the notational distinction between a random variable and its observed value. An observable random variable (either scalar or vector) will always be denoted by a capital letter, for example, $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$; its observed value will be denoted using small case, for example, $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$.

In recalling the classical definition of confidence interval, let $\vec{X}=\{X_1, X_2, \ldots, X_n\}$ be a random sample of size *n*, and θ be a scalar parameter of interest. It should be noted that θ may actually be a function of other parameters. If there exists statistics $L(\vec{X})$ and $U(\vec{X})$ whose distributions are free of any unknown parameters, and that satisfy $1-\alpha=P(L(\vec{X})<\theta<U(\vec{X}))$, then a $(1-\alpha)100\%$ confidence interval for θ is given by $(L(\vec{x}), U(\vec{x}))$.

One technique that is used to construct a classical confidence interval for a parameter θ is the pivotal method. It is comprised of the following four steps: 1) Obtain a pivot, that is, a random quantity whose distribution does not depend upon any unknown parameters. 2) Write a simple probability statement that bounds the pivot. 3) Invert this into a probability statement that bounds θ. 4) If the bounds for θ do not depend on any unknown parameters, one can use them to obtain a confidence interval for θ.

We note that a pivot is a function of:

a) the random data (typically a set of sufficient statistics),
b) the unknown parameter of interest, and
c) perhaps other nuisance parameters.

As long as the cumulative distribution function associated with the observations is continuous, a pivot will exist (Mood, Graybill & Boes, 1974). However, for many problems, no pivot exists which can be inverted to form a confidence interval. In many cases this is due to the presence of nuisance parameters. In such instances, approximate methods, or some other technique for constructing a confidence interval is needed.

One relatively new technique involves the use of generalized pivots (Weerahandi, 1993). A generalized pivot is a function of the same three arguments as a traditional pivot *plus* the observed data. A generalized pivot can be written as $R(\vec{X}, \vec{x}, \theta, \eta)$, reminding us of its four arguments; and it must satisfying the following conditions:

a) the distribution of $R(\vec{X}, \vec{x}, \theta, \eta)$ is free of any unknown parameters, and
b) the observed value $r=R(\vec{x}, \vec{x}, \theta, \eta)$ is free of the nuisance parameter η.

Furthermore, we say that $R(\bar{X}, \bar{x}, \theta, \eta)$ is a generalized pivot for θ if $r = \theta$. The percentiles of a generalized pivot for θ yield generalized confidence limits/bounds for θ.

Constructing Generalized Pivots

Since their introduction, generalized confidence intervals have been utilized on a limited basis, in part because of the lack of a clear method for constructing generalized pivots. Even in his seminal paper Weerahandi wrote "The problem of finding an appropriate pivotal quantity is a nontrivial task", and "… the construction of pivotals requires some intuition." Recently, a seven-step algorithm has been proposed (Iyer & Patterson, unpub.) that elucidates how generalized pivots and confidence intervals can be obtained. Their algorithm is given here, along with its implementation towards an interval estimate for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$, the probability of non-perforation.

Step 1: Find a set of independent, sufficient statistics for the sample.

Assuming a normal population, the sample mean, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and the sample variance, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ are sufficient statistics.

Step 2: From these, find a same-sized set of statistics whose distributions are independent of the unknown parameters.

We have $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ and $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

Step 3: Solve for the unknown parameters in terms of the statistics in Step 2.

With some simple algebraic manipulation, one obtains $\mu = \bar{X} - ZS\sqrt{\frac{n-1}{nV}}$ and $\sigma = S\sqrt{\frac{n-1}{V}}$.

Step 4: Substitute the expressions for the unknown parameters in Step 3 into θ.

Starting with the parameter of interest, θ, we make substitutions for μ and σ, then simplify:

$$\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right) = \Phi\left(\frac{x_0 - \left(\bar{X} - ZS\sqrt{\frac{n-1}{nV}}\right)}{S\sqrt{\frac{n-1}{V}}}\right) = \Phi\left(\frac{x_0 - \bar{X}}{S\sqrt{\frac{n-1}{V}}} + \frac{Z}{\sqrt{n}}\right).$$

Step 5: Substitue the (random) sufficient statistics with their observed values.

In the previous expression for θ, the sufficient statistics appear in the first addend. After substituting the observed values, we have the random variable $\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{\frac{n-1}{V}}}+\frac{Z}{\sqrt{n}}\right)$. Notice that because of the substitution, this expression is no longer equated to θ. But more importantly, note that this random variable has a distribution that is independent of either μ or σ^2.

Step 6: Substitute the remaining random terms with their sufficient-statistic based equivalents. Finally, this is the generalized pivot for θ.

Denoting this generalized pivot by *R*, one obtains $R=\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}}+\frac{1}{\sqrt{n}}\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)$. R meets the definition of a generalized pivot for θ since it has the same parameter-free distribution as $\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{\frac{n-1}{V}}}+\frac{Z}{\sqrt{n}}\right)$, and its observed value is

$$r=\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{n-1}}\sqrt{\frac{(n-1)s^2}{\sigma^2}}+\frac{1}{\sqrt{n}}\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)=\Phi\left(\frac{x_0-\mu}{\sigma}\right)=\theta .$$

Step 7: The percentiles of the generalized pivot form generalized confidence limits (or confidence bounds) for θ.

In general, these percentiles may be obtained through Monte-Carlo simulation. For example, we know that $R=\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}}+\frac{1}{\sqrt{n}}\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)$ has the same distribution as $\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{\frac{n-1}{V}}}+\frac{Z}{\sqrt{n}}\right)$. Therefore, one could generate, and order a "large" sample of values of *R*, e.g., $R_{(1)}, R_{(2)}, \ldots, R_{(10000)}$. Then a 95% two-tailed generalized confidence interval is $\left(\frac{R_{(250)}+R_{(251)}}{2}, \frac{R_{(9750)}+R_{(9751)}}{2}\right)$.

Equating the Generalized and Classical Confidence Intervals for Probability of Non-Perforation

In some cases, the percentiles of the generalized pivot may be expressed in closed form. When a conventional confidence interval exists (for example, a confidence interval for the mean of a normal random variable), the generalized confidence interval will reduce to it. In the following exercise, we will show how the generalized confidence interval for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$ is equivalent to the interval derived by Owen & Hua.

Consider a $(1-\alpha)100\%$ lower confidence bound for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$. It is defined in Step 7 to be that value, B_L, for which $1-\alpha = P(B_L \le R)$. Recalling the distributional equivalent of R discussed in Step 6, we have.

$$1-\alpha = P\left(B_L \le \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{\frac{n-1}{V}}} + \frac{Z}{\sqrt{n}}\right)\right).$$

Algebraically manipulating this equation, one obtains

$$1-\alpha = P\left(\frac{\bar{x} - x_0}{s/\sqrt{n}} \le \frac{Z - \sqrt{n}\Phi^{-1}(B_L)}{\sqrt{V/n-1}}\right)$$

Notice that the right side of the inequality is a non-central t random variable with non-centrality parameter $-\sqrt{n}\Phi^{-1}(B_L)$ and $n-1$ degrees of freedom. However, a non-central t random variable with non-centrality parameter $-\sqrt{n}\Phi^{-1}(B_L)$ is the mirror image of a non-central t random variable with non-centrality parameter $\sqrt{n}\Phi^{-1}(B_L)$ (Johnson and Kotz, 1970). Therefore,

$$1-\alpha = P\left(T_{n-1,\sqrt{n}\Phi^{-1}(B_L)} \le \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right).$$

But this probability is equivalent to the cumulative distribution function of a non-central t random variable with non-centrality parameter $\sqrt{n}\Phi^{-1}(B_L)$ and $n-1$ degrees of freedom, evaluated at $\frac{x_0 - \bar{x}}{s/\sqrt{n}}$. This is the same result as proven by Owen and Hua. The value of the non-centrality parameter that satisfies this probability statement (and in turn produces the desired lower confidence bound, B_L) must be solved using numerical methods, e.g., the bisection method.

Application

Fifteen projectiles ($n = 15$) are fired into armor plates to record the depth of penetration. The results in ascending order are given in the table below. Find an estimate for the probability of non-perforation if the production armor is to be 60 units thick. (Note: these data are not actual, but have been contrived for illustrative purposes.)

Sample Data

29.4	46.1	47.5	52.7	57.6
34.6	46.2	50.9	55.9	60.8
43.3	46.4	52.7	56.4	69.5

The sample mean is $\bar{x} = 50$ and sample standard deviation is $s = 10$. Therefore, the plug-in point estimate is $P(X < 60) = \Phi\left(\frac{60 - \bar{x}}{s}\right) = \Phi\left(\frac{60 - 50}{10}\right) = \Phi(1) = .841$.

A 95% generalized confidence interval is (B_L, B_U) where B_L satisfies $.975 = P\left(T_{14,\sqrt{15}\Phi^{-1}(B_L)} \leq \frac{60 - 50}{10/\sqrt{15}}\right) = P\left(T_{14,\sqrt{15}\Phi^{-1}(B_L)} \leq \sqrt{15}\right)$ and B_U satisfies $.025 = P\left(T_{14,\sqrt{15}\Phi^{-1}(B_L)} \leq \sqrt{15}\right)$. Using the bisection method in MATLAB®, we obtain $B_L = .642$ and $B_U = .947$.

By allowing the thickness of the armor to vary, one can generate confidence bands for the normal cumulative distribution function (see Table 1). Perhaps of more importance to the armor design engineer would be a plot that shows the relationship between armor thickness and a one-sided lower confidence bound for the probability of non-perforation (see Table 2). Such a chart would allow the engineer to choose that thickness which offers the desired level of protection against perforation by an enemy projectile.

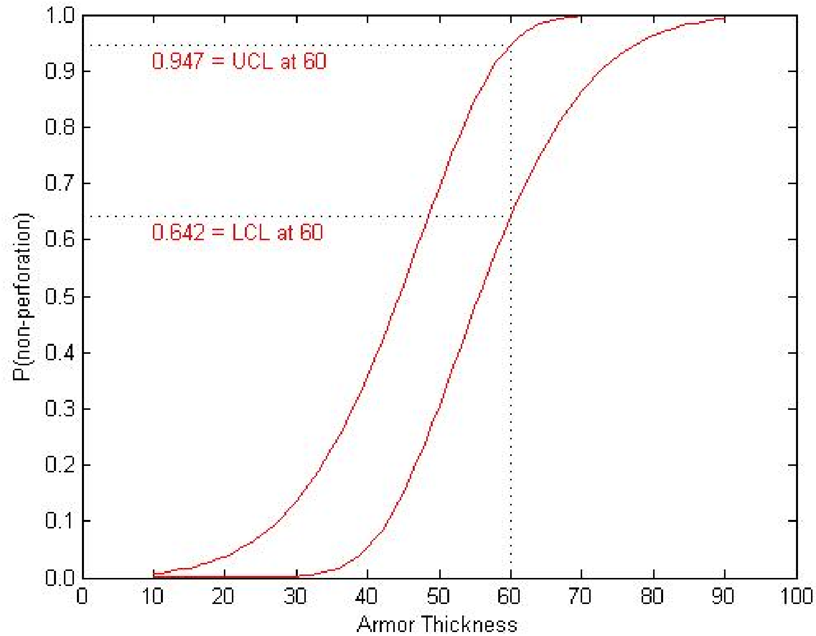

Figure 1. 95% confidence bands for the probability of non-perforation.

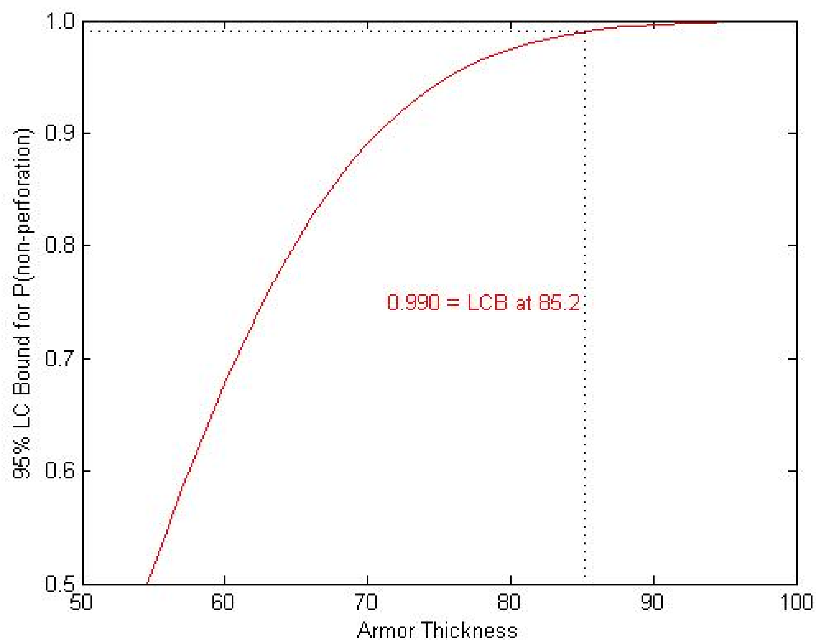

Figure 2. Armor thickness versus 95% lower confidence bound for the probability of non-perforation. For example, a thickness of 85.2 yields a high level (at least 99%) of protection against perforation.

Concluding Remarks

The beauty of this theory is that a generalized confidence interval can be constructed for any function of the normal parameters. In particular, if the parameter of interest is some function of the normal parameters, $\theta = g(\mu, \sigma)$, then it can be shown that a generalized pivot for θ is $R = g\left(\bar{x} - (\bar{X} - \mu)\frac{s}{S}, \frac{s\sigma}{S}\right)$. For example, Weerahandi (2004) discusses estimation of $\theta = \frac{\mu + \sigma}{\mu^2 + \sigma^2}$; and Iyer and Patterson (unpub.) derive interval estimates for $\theta = P(a < X < b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$.

On the surface, it may appear that generalized confidence intervals hold great potential for solving complex estimation problems. However, it is important to understand that the actual coverage probability of a generalized confidence interval may not necessarily equal $1 - \alpha$ when the percentiles of the pivot have no closed-form solution. The actual coverage probability may be influenced by nuisance parameters. A detailed simulation study is necessary to evaluate the (approximate) coverage probability, and hence the quality of the generalized confidence interval.

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