On Computing and Comparing the Reliability of Competing Networks [1]

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Abstract

Comparing the reliability of two networks of more than modest size can be a computationally intensive exercise. In this paper, domination theory and the notion of the signature of a network, and their respective roles in calculating the reliability of a network, are briefly reviewed. The computational advantages of the former, and the interpretive richness of the latter, beg the question: how are the two related? The exact functional relationship between the signature vector and the vector of signed dominations is obtained. A detailed example is given in which the connection between these two concepts is usefully exploited.

A network G with v vertices and n edges is typically denoted by the symbol $G(v, n)$. We follow the usual convention that postulates that nodes cannot fail, but that edges can be in either a functioning or a failed state. In communication networks, as in many other types of networks, the primary quality characteristic of interest is connectivity. A two-terminal network is connected if there is at least one set of functioning edges providing a path from one terminal to the other. We will restrict attention to networks that are coherent, that is, to networks with the property that every edge is relevant and that all supersets of path sets are also path sets.

The network characteristics upon which we will be focusing are defined in terms of edges whose lifetimes are treated as independent and identically distributed random variables. We will be concerned with the distribution of T, the failure time of the network and, in particular, with the probability that it is connected at a given time t_0. In the latter instance, we'll treat the states of edges (i.e. working or failed states) as independent Bernoulli variables. For concreteness, all references in the sequel to the reliability of a network pertain to two-terminal reliability.

It is known, of course, that the reliability of the network $G(v, n)$ in i.i.d. edges can be expressed as a polynomial $h(p)$ of order n, that is, as

$$h(p) = \sum_{r=1}^{n} d_r p^r, \tag{1}$$

where p is the common success probability for the edges. Satyarananaya and Prabhakar [10] showed that the coefficients in (1) could be obtained as the signed dominations associated with the network. Indeed, domination theory, described in 1984 as a breakthrough by Agrawal and Barlow [1] among computational tools in network reliability, continues to be a widely used algorithmic vehicle for calculating the reliability polynomial. We review the concept of dominations in Section 2. We note that, as useful as domination theory has proven to be in simplifying the computation of the reliability of a network, it has not been found particularly useful in

[1] Work supported in part by grant DAAD 19-99-1-1082 from the U.S. Army Research Office.

comparing one network design with another as is required, for example, in searching for universally optimal networks of a given size (v, n).

A quite different tool was introduced by Samaniego [9] for studying the performance properties of coherent systems. The concept of signature applies equally well to network reliability. The signature of a network is a probability vector $\boldsymbol{s}$ whose components are simply the respective probabilities that the first, second, $\cdots$, and nth edge failures (ordered by time of occurrence) are fatal to the network. Assuming, again, i.i.d. edge states at a fixed time t_0, the reliability polynomial of a network can be expressed in terms of the network's signature vector. Unlike the domination vector, the properties of the signature vector are readily interpretable and have a close relationship to the failure time T of the network itself. We review the notion of signature, and some of the problems to which it has been applied, in Section 3.

The main goal of this paper is to identify the exact relationship between the vector of signed dominations $\boldsymbol{d}$ and the signature vector $\boldsymbol{s}$. This is accomplished in Section 4. Because dominations are central to the computation of the reliability of a network, and signatures are rich in interpretation regarding the relative performance of competing networks, the exact linkage of the two through the functional relationship $\boldsymbol{s} = f(\boldsymbol{d})$ established here enables one to exploit the benefits of both. Our closing example demonstrates the utility of this linkage.

2. A Brief Look at Domination Theory

The notion of dominations was discovered in the process of seeking a reduction in the complexity of the well-known inclusion-exclusion formula (see [6]) for calculating the probability that all edges are functioning in at least one of a given network's minimal path sets. The inclusion-exclusion rule applies to the union of any m sets, and may be written as

$$\begin{aligned} P\left(\bigcup_{i=1}^{m} A_i\right) &= \sum_1 P(A_i) - \sum_2 P(A_i \cap A_j) + \sum_3 P(A_i \cap A_j \cap A_k) \\ &- \cdots + (-1)^{m+1} P\left(\bigcap_{i=1}^{m} A_i\right), \end{aligned} \tag{2}$$

where $\sum_i$ represents a sum over all i-fold intersections. If A_i in (2) represents the event that all edges in the ith minimal path set are working, and there are m minimal path sets in all, then the formula in (2) provides the probability that the network will function.

Suppose that one has a list of minimal path sets of a given network in n i.i.d. edges. A formation is defined as a union of minimal path sets. A *formation* is thus the union of the edges in a fixed collection of minimal path sets. Finally, an *i-formation* is a union of the components in a set of i minimal path sets. For example, the union $\{1, 2, 3, 4\}$ of the minimal path sets $\{1, 2\}$, $\{2, 3\}$, and $\{3, 4\}$ would be an example of a formation that is both a 2-formation and a 3-formation. We will refer to a particular formation as even if it is the union of an even number of minimal path sets and as odd if it is the union of an odd number of minimal path sets. It can, of course, be both at the same time.

The minimal path sets of this network are the sets of edges $\{1, 4\}$, $\{2, 5\}$, $\{1, 3, 5\}$, and $\{2, 3, 4\}$. The *signed domination* of a given union of minimal path sets is simply the difference between the number of even dominations and the number of odd dominations for that union. Satyarananaya and his co-workers showed that is a large variety of network reliability settings, the awkward expression for network reliability in (2) could be replaced by the simple form of the reliability polynomial in (1), where $(d_1 ... d_n)$ is the vector of signed dominations.

3. A Brief Look at Signatures.

The signature of a network of order n (that is, having n edges) is defined as the probability distribution $\boldsymbol{s}$ on the integers $\{1, 2, \cdots, n\}$ for which

$$s_i = P(T = X_{(i)}), \quad i = 1, 2, \cdots, n, \tag{3}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics from a random (i.i.d.) sample drawn from the (arbitrary) continuous lifetime distribution F, and T is the lifetime of the network.

The fact that the signature $\boldsymbol{s}$ depends only on the network design, and not on the distribution F, is a consequence of the fact that each of the $n!$ orderings of the failure times $X_1, X_2, \cdots, X_n$ of the n edges is equally likely to occur under the i.i.d. assumption. Thus, the probability that the ith edge failure is fatal to the network is solely dependent on the likelihood that the last working edge in some minimal cut set is the ith edge to fail overall. In other words, calculating s_i is simply a matter of examining minimal cut sets and counting how many among the equally likely permutations of $X_1, X_2, \cdots, X_n$ coincide precisely with a particular minimal cut set failing, before any other, upon the occurrence of $X_{(i)}$, the (ordered) ith edge failure time.

As shown in Samaniego [9] (see also [7]), the survival function of a network's lifetime T can be written as a simple function of $\boldsymbol{s}$ and F. When focusing on the reliability of the network at a fixed time t_0, where $P(X_j > t_0) = p$ for all j, this representation reduces to the reliability polynomial in pq-form, that is, in the form

$$h(p) = \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j}. \tag{4}$$

The tail probabilities of the signature vector $\boldsymbol{s}$ have an interpretation through the concept of path set. This connection was noted by Boland [4] and exploited in his study of indirect majority systems. As is apparent from (4), the coefficient of $p^j q^{n-j}$ in the reliability polynomial in pq-form can be interpreted as the number of path sets of order j, as it is precisely those sets, among the collection of $\binom{n}{j}$ sets with exactly j working components, that each contribute the positive probability $p^j q^{n-j}$ to the reliability polynomial. If we let a_j stand for the proportion of path sets among the $\binom{n}{j}$ sets of j working components (with the complementary components non-working), then we see that the reliability polynomial can be written as

$$h(p) = \sum_{j=1}^{n} a_j \binom{n}{j} p^j q^{n-j}. \tag{5}$$

It follows that the vector $\boldsymbol{a}$, which is fundamentally related to path sets, and the vector $\boldsymbol{s}$, which is fundamentally related to cut sets, are related to each other through the system of equations

$$a_j = \sum_{i=n-j+1}^{n} s_i, \quad j = 1, \cdots, n, \tag{6}$$

For future reference, the linear relationship between the vectors $\boldsymbol{a}$ and $\boldsymbol{s}$ will be denoted as $\boldsymbol{a} = \boldsymbol{P}\boldsymbol{s}$.

In the introduction, we alluded to the fact that signatures are rich in interpretation and are particularly useful in the comparison of competing networks. We summarize here a collection of results that support this remark. The random variables

X_1 and X_2, discrete or continuous, are stochastically ordered (i.e., $X_1 \leq_{st} X_2$) if the survival functions $S_i(x) = P(X_i > x)$ are suitably ordered, that is, if $S_1(x) \leq S_2(x)$ for all x. We say that X_1 is smaller than X_2 in the hazard rate (or uniform stochastic) ordering if the ratio of survival functions $S_2(x)/S_1(x)$ is nondecreasing in x. This ordering will be denoted by $X_1 \leq_{hr} X_2$. Finally, X_1 is said to be smaller than X_2 in the likelihood ratio ordering ($X_1 \leq_{lr} X_2$) if the ratio $f_2(x)/f_1(x)$ is nondecreasing in x, where f_i represents the density or probability mass function of X_i. As is well known, stochastic order is the weakest of these three relations; indeed, it is easy to verify that $lr \Rightarrow hr \Rightarrow st$. With the notation established above, we may now restate results from Kochar, Mukerjee and Samaniego [7] relating properties of signatures to properties of network lifetimes.

Theorem 1 *Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of two networks with n i.i.d. edges, and let T_1 and T_2 be their respective lifetimes. If $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$ or $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$ or $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$, then $T_1 \leq_{st} T_2$ or $T_1 \leq_{hr} T_2$ or $T_1 \leq_{lr} T_2$, respectively.*

The results above have been applied with profit to stochastic comparisons of k-out-of-n structures with system-wise or component-wise redundancy (see [7]), to indirect majority systems of varying design (see [4]) and to consecutive k-out-of-n systems with varying n (see [5]). We will apply these results again in an example in the concluding section.

4. The Linkage Between Dominations and Signatures.

As noted above, the reliability polynomial of a network with signature $\boldsymbol{s}$ may be written as

$$h(p) = \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j} \tag{7}$$

or equivalently as

$$h = (p) \sum_{r=1}^{n} \left(\sum_{j=1}^{r} a_j \binom{n}{j} \binom{n-j}{r-j} (-1)^{r-j} \right) p^r, \tag{8}$$

where $\boldsymbol{a}$ is as in (6). Examining the expressions in (1) and (8), we see that the vectors $\boldsymbol{d}$ and $\boldsymbol{a}$ are related via the equations

$$d_r = \sum_{j=1}^{r} a_j \binom{n}{j} \binom{n-j}{r-j} (-1)^{r-j}, \qquad r = 1, \cdots n. \tag{9}$$

Alternatively, the components of the domination and signature vectors satisfy the relationships

$$d_r = \sum_{j=1}^{r} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} \binom{n-j}{r-j} (-1)^{r-j}, \qquad r = 1, \cdots, n. \tag{10}$$

If we denote the linear relationship between $\boldsymbol{d}$ and $\boldsymbol{a}$ in (9) as $\boldsymbol{d} = \boldsymbol{M}a$ and if we denote the linear relationship between $\boldsymbol{a}$ and $\boldsymbol{s}$ in (6) as $\boldsymbol{a} = \boldsymbol{P}s$, then we may write the relationship of interest as $\boldsymbol{s} = \boldsymbol{P}^{-1}\boldsymbol{M}^{-1}\boldsymbol{d}$.

The following result identifies this latter relationship explicitly.

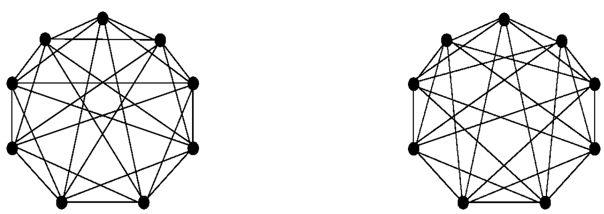
Theorem 2 *Let $\boldsymbol{d}$ and $\boldsymbol{s}$ denote the domination and signature vectors for a given network of order n. Then for $i = 1, \cdots, n$ we have*

$$s_i = \sum_{j=1}^{n-i} \frac{-(n-i)_j + (n-i+1)_j}{(n)_j} d_j + \frac{(n-i+1)_{n-i+1}}{(n)_{n-i+1}} d_{n-i+1}, \tag{11}$$

where $(k)_j$ denotes the number of ways of selecting without replacement j items from k items, accounting for the order of selection. For a proof of this result, see Boland, Samaniego and Vestrup [6].

We suggested in Section 1 that the comparison of networks via their vector of dominations was unintuitive and, for complex networks, quite difficult. The reason for this is that the difference of two polynomials in standard form (that is, in the form displayed by (1)) is another polynomial in standard form. For two complex networks, the difference polynomial $\sum(d_{2r} - d_{1r})p^r$ will typically be of quite high degree. Because of the requirement $\sum d_r = 1$ on the domination vector of an arbitrary network, it can never be the case that all coefficients of the difference polynomial will have the same sign. Thus, determining whether one reliability polynomial is uniformly larger than another for all $0 < p < 1$ is a task equivalent to finding the roots of a high degree polynomial. But that algebraic problem is a quite famous one, a problem that was dramatically resolved by Evariste Galois. Finding roots of polynomials of degree greater than 4 is a problem that is not "solvable by radicals"; thus, closed-form expressions for the solutions of such problems are not possible in general.

Transforming this problem into the world of signatures changes things substantially. To see this more graphically, let us consider the comparison between the two G(9,27) networks of order, pictured below.



AX3 AX4

In standard form, the reliability polynomials for these two networks are found to be

$$\begin{aligned} h_{AX3}(p) = \quad & 419904p^{27} - 6021144p^{26} + 41705280p^{25} - 18489826p^{24} \\ & +586821717p^{23} - 1413876060p^{22} + 2677774329p^{21} \\ & -4074363810p^{20} + 5048856414p^{19} - 5135792742p^{18} \\ & +4303029693p^{17} - 2967712776p^{16} + 1676975886p^{15} \\ & -769265910p^{14} - 282176568p^{13} + 80853282p^{12} \\ & +17445456p^{11} - 2667060p^{10} + 257634p^{9} - 11828p^{8} \\ h_{AX3}(p) = \quad & 414720p^{27} - 5934288p^{26} + 41015964p^{25} - 181453380p^{24} \\ & +574666025p^{23} - 1381692972p^{22} + 2611463517p^{21} \\ & -3965536554p^{20} - 4904464002p^{19} + 4979513718p^{18} \\ & +4164454729p^{17} - 2867022480p^{16} + 1617256842p^{15} \\ & -740601350p^{14} - 271201476p^{13} + 77576922p^{12} \\ & +16709916p^{11} - 2550156p^{10} + 245898p^{9} - 11268p^{8} \end{aligned}$$

While the uniform superiority of one of these networks over the other is certainly not obvious by inspection, one could by numerical means show that $h_{AX3}(p) \geq h_{AX4}(p)$ for all $p \in [0, 1]$. However, the comparison of the two signatures immediately yields this same conclusion, and in addition, a stronger one. From table 1 below, it is apparent that the signature of network AX3 is stochasitcally larger than that of AX4, yielding the uniform domination of AX3 over AX4 alluded to above. However, the comparison of the two signatures vectors yields an additional new insight. The ratios of the two survival functions displayed in the last column of table 1 shows that AX3 dominates AX4 in the hazard rate ordering as well. We can thus rightly say that AX3 is not only better than AX4, it is actually quite a bit better!

TABLE 1: Signature Tail Probabilities $S(x) = \sum_{i=x}^{27} s_i$ And Their Ratios

x	$S_{AX3}(x)$	$S_{AX4}(x)$	$S_{AX3}(x)/S_{AX3}(x)$
1	1.0	1.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	1.0	1.0	1.0
5	1.0	1.0	1.0
6	1.0	1.0	1.0
6	1.0	1.0	1.0
7	0.999970	.0999970	1.0
8	0.999787	.0999787	1.0
9	0.999149	0.999149	1.0
10	0.997367	0.997367	1.0
11	0.993612	0.993612	1.0
12	0.985922	0.985922	1.0
13	0.971744	0.971743	1.0000005
14	0.947220	0.947214	1.0000063
15	0.906907	0.906867	1.0000442
16	0.843421	0.843240	1.0002148
17	0.747317	0.746717	1.0008024
18	0.607883	0.606416	1.0024183
19	0.417560	0.415077	1.0059834
20	0.189140	0.186804	1.0125000
21	0.0	0.0	–
22	0.0	0.0	–
23	0.0	0.0	–
24	0.0	0.0	–
25	0.0	0.0	–
26	0.0	0.0	–
27	0.0	0.0	–

The main thesis of this note can be summarized quite succinctly: Domination theory is a useful tool in making network reliability calculations. However, for the purpose of comparing the performance characteristics of two competing networks, reliabilities expressed in terms of signed dominations will tend to have little intuitive content and may be of limited use (except for the possibility of brute force computation). The utility of signatures in the comparison of networks of the same size immediately raises questions about the exact relationship between the domination and signature vectors. The functional relationship linking the signature vector with the vector of dominations is displayed above. This linkage allows one to combine the computational advantages of domination theory with the intuitive and interpretive qualities of signatures for the purpose of making comparisons among networks. The growing but still inconclusive literature on the existence, uniqueness and identification of uniformly optimal networks of a given size (see, for example, [2], [3], [8], and [12]) should benefit from the application of these linked tools.

5. References.

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